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Computing unsegmented product maps from preference data by means of single ideal point models is commonly thought to be impossible because of indeterminacy problems. The authors show that this mathematical indeterminacy can be overcome by incorporating dependent sampling assumptions into a probabilistic multidimensional scaling (MDS) model. As a result, product space maps can be estimated for single markets from preference data alone. If desired, dissimilarity data can be combined with preference data to produce jointly estimated product space maps. The authors illustrate the advantages of the proposed approach with real and simulated data. They also make comparisons to both internal and external deterministic models. The results are favorable.

A Single Ideal Point Model for Market Structure Analysis

We define market structure analysis as a process for representing the interrelationships of a set of products or brands in a way that reflects consumers' evaluations of the items in the set (Elrod 1991; Grover and Dillon 1985). The set may consist of competing brands in a particular product category or cross category products that are being considered for a particular occasion, such as choosing an entertainment alternative, selecting a residence, deciding where to go on vacation, or purchasing a gift (Bettman and Sujan 1987; Johnson 1989).

Ideal point models are commonly used to describe market structures in which one or more products can have too much or too little of at least one characteristic—a beverage may be too sweet or not sweet enough, a vacation alternative may be too "posh" or not "posh" enough. Ideal point models are spatial models in which the preference of a consumer or consumer segment for a product is an inverse function of the distance between the point that represents the product and the ideal point that represents the consumer (Carroll 1972; DeSarbo, De Soete, and Eliashberg 1987; DeSarbo et al. 1990; DeSarbo and Rao 1986; De Soete, Carroll, and DeSarbo, 1986; Elrod 1988; Gaul 1989; Green, Carmone, and Smith 1989; Jedidi and DeSarbo 1991; Kamakura and Srivastava 1986; MacKay and Dröge 1990; Zinnes and MacKay 1992). Like other spatial models, ideal point models assume that the coordinate dimensions are continuous variables. Aspatial models, which typically represent brands and customers as terminal nodes of some tree structure, provide an alternative approach to market structure analysis (for a recent review of aspatial models, see DeSarbo, Manrai, and Manrai 1993).

Sometimes there is a need to estimate a market or market segment structure with a single ideal point. This need occurs when a single ideal point is desired for an entire market or when a market segment must be evaluated by itself, because the other segments do not perceive the products in the same or closely related ways. For example, a comparison of the home, business, and educational markets for personal computers, which involves three ideal points, may be represented best by three separate single ideal point analyses if the three segments perceive the interrelationships among the products very differently. When there is little discernible difference in how market segments perceive products, then a single solution with multiple ideal points should be favored over multiple solutions with single ideal points because of the single solution's greater simplicity.

It is relatively easy to estimate the location of a single ideal point using an external analysis in which the points defining the products are assumed to be known before the analysis begins. This typically happens when similarity data have been analyzed and perceptual maps of the products have been obtained. When this has not been done and preference data only are available, an internal analysis is required. Then, it is necessary to estimate simultaneously the locations of both the ideal points and the product points from only the preference data.

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When similarity data are available, an internal analysis may still be desired if it is thought that the attributes on which preference judgments are made differ from those on which similarity judgments are made. Lefkoff-Hagius and Mason (1993) provide evidence that this may indeed be the case and report that beneficial attributes are relatively more important in preference judgments and characteristic attributes are relatively more important in similarity judgments.

Estimation of a single ideal point using an internal analysis encounters severe indeterminacy problems. There appear to be insufficient constraints for determining the position of the ideal point relative to the products. Schönemann and Wang (1972), describing their metric unfolding model, state that in a p dimensional space, a minimum of $p + 1$ ideal points is needed to obtain a unique solution. Coxon (1982), referring to his experience with nonmetric unfolding models, recommends at least 30 ideal points for a two-dimensional solution. Anderson (1981, p. 369) states that one must have "ideal points distributed across the stimulus range ... a [single ideal point] analysis is not possible."

We show that by adding dependent sampling assumptions to a recently developed probabilistic unfolding model (MacKay and Zinnes 1995), it *is* possible to estimate a single ideal point model when using an internal analysis that is based on preference data only. We start by describing the nature of this indeterminacy problem more fully. This background section shows why probabilistic ideal point models incorporating dependent sampling assumptions offer a solution to the indeterminacy problem. We next describe how dependent sampling can be built into a probabilistic model. In addition to making the estimation of a single ideal point model possible, dependent sampling enables the variances of actual and ideal products to be uniquely identified. In the subsequent sections, we use (1) simulated data to illustrate an internal analysis and (2) real data from an intercultural study of American and Japanese markets to illustrate the simultaneous analysis of preference and similarity data. We also explicitly test the assumption of equivalent product spaces underlying both the preference and similarity data. Finally, we compare the results from the proposed probabilistic model to those obtained from popular deterministic models with both the simulated and empirical data.

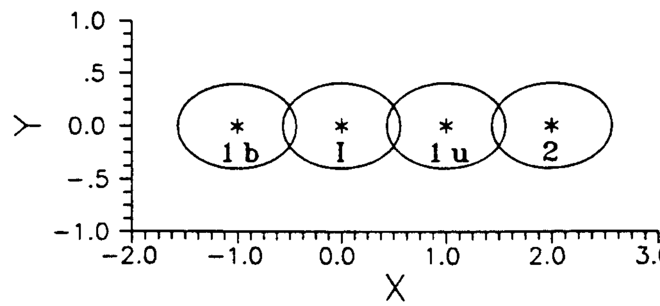
BACKGROUND

Unidimensional unfolding (Coombs 1950) assumes that if a consumer is asked which of two products is preferred, the product that is nearer the consumer's ideal position will be chosen. Coombs referred to the joint distribution, or continuum of stimulus and ideal points, as a *J scale*. If, for any single consumer, the J scale is folded over the ideal point, then an individual distribution, or *I scale*, results, in which the ordering of the points corresponds to the consumer's relative preference for the products. Multidimensional unfolding (Bennett and Hays 1960), which occurs when the points representing the products and consumers are in a multidimensional space, is based on similar assumptions.

Early unfolding models were deterministic. The standard deviations of the points representing the products and consumers were assumed implicitly to be zero. To estimate a J scale from preference or choice data with a deterministic model, a researcher must have multiple ideal points. For a single ideal point, (1) multiple, equally valid J scales can be derived from an I scale and (2) indeterminacy problems occur. In the unidimensional case, for example, an I scale with values of 1 and 2 for two products can be equally well represented by a unilateral J scale with the ideal point at the origin and the product points at 1 and 2 or a bilateral J scale with the ideal point at the origin and the product points at −1 and 2.

Figure 1

BILATERAL AND UNILATERAL POINTS



Note: Bilateral points {1b,I,2}.
Unilateral points {I,1u,2}.
The ellipses indicate the size of the standard deviations on each axis.

More recently, probabilistic unfolding models, in which some or all of the points are treated as random variables with non-zero variances, have been proposed (for reviews, see Bossuyt 1990; De Soete, Feger, and Klauer 1989; Elrod 1991). A probabilistic, two-dimensional example, corresponding to the unilateral and bilateral scales mentioned previously, is shown in Figure 1. In this figure, I represents a single ideal point, 1b and 1u represent alternative locations for one product, and 2 represents the location of a second product. The product pair {1u, 2} is a unilateral pair relative to I, whereas the product pair {1b, 2} is a bilateral pair relative to this same ideal point.

In Figure 1, the distributions of the points, assumed to be bivariate normal, are represented by ellipses. All points on an ellipse have the same likelihood or probability density. Following the terminology of Thurstone (1927), Figure 1 describes a Case 5 situation in which the standard deviations of all points on any given dimension are the same. However, the standard deviations of the points in this figure are not the same on both axes. Thus, the points are located in an *anisotropic* space, that is, the properties of the space are not the same in all directions. Circular ellipses would indicate that the points were located in an *isotropic* space, that is, the properties of the spaces are the same in all directions.

Isotropic spaces are easier to deal with mathematically than anisotropic spaces, but anisotropic spaces seem more reasonable for products and other marketing stimuli. Some attributes, such as the price of a new product, may be well-known and characterized by a small standard deviation, whereas other attributes, such as anticipated satisfaction, may be poorly known and characterized by a large standard deviation. Even unidentified physical stimuli, such as the products in a blind taste-test, are likely to have anisotropic distributions because of the differential sensitivity of chemosensory receptors.

Another advantage of anisotropic models is that they are characterized by moderate stochastic transitivity—even in the Case 5 situation. In contrast, Case 5 isotropic models possess strong stochastic transitivity, a condition that is frequently violated in empirical examples. (For a discussion of the implications of different types of stochastic transitivity, see De Soete and Carroll 1992.)

Figure 1 provides a simple test case for an internal unfolding model with a single ideal point. The question we ask here is: Using just pair-wise preference judgments, is it possible to determine if 1b or 1u is the true location of the first product?

There are two primary reasons why probabilistic unfolding models are of special interest in modeling preferences characterized by a single ideal point. First, because we are estimating only one ideal point, the magnitudes of all the variances—the ideal as well as the actual products—are likely to be high. Under these conditions, deterministic models produce large, systematic biases. A primary cause of this bias is the confounding of distance and variation that occurs with deterministic models. When, for example, a consumer estimates the distance of an actual product from his or her ideal product, the distance will, assuming a nonzero variance for one of the distributions, always be positive. This occurs even when the centroids of the actual and ideal products are exactly the same. Deterministic models cannot distinguish separation caused by random sampling from separation caused by the distance between centroids. In many cases, the separation of items in a deterministic product map may be due more to variance in perception than to distances between centroids.

Second, as noted previously, when only a single ideal point is used, indeterminacy problems are likely to occur with a deterministic model. However, when probabilistic models are used, it is possible to overcome both the bias and indeterminacy problems and determine, from an internal analysis of only preference data, the location of the single ideal product, as well as the locations of the actual products.

Systematic bias and indeterminacy are issues that should concern any practitioner. Their presence will cause estimates of the market structure that are meaningless. To make matters worse, systematic bias and indeterminacy often go undetected because they can occur even when the deterministic model fits the original data well.

In Elrod's (1991) recent review of internal structure models, the only model mentioned that can evaluate both the ideal points and stimulus points, when both are treated as random variables, is MacKay and Zinnes's (1986) PROSCAL model. When variability in product perception exists, it is desirable to distinguish it from variability in perceiving the attributes of an ideal product. The managerial action required to change the variability in how an actual product is perceived may be different from that required to change the variability in what is looked for in an ideal product.

The PROSCAL model is a conceptually simple model that combines a multidimensional generalization of Thurstone's (1927) pair comparison model with Coombs's (1950) unfolding model. Specifically, for each actual or ideal product $\mathbf{P}_j$, $j = 1,...,m$, there is a corresponding p-dimensional random vector $\mathbf{X}_j = (X_{j1},...,X_{jp})$ that has a p-variate normal distribution with mean vector $\mu_j = (\mu_{j1},...,\mu_{jp})$ and covariance matrix Σ_j Judgments or choices of consumers are assumed to be based on values sampled from the $\mathbf{X}_1 ... \mathbf{X}_m$ distributions. If a consumer has a precise, consistent image of a product, be it actual or ideal, then we expect the diagonal elements (i.e., the variances) of Σ_j to be small. On the other hand, if a consumer has a vague image, we expect the diagonal elements of Σ_j to be large.

Data for the PROSCAL model consist of preference ratio judgments. These judgments require each consumer to indicate which of two products he or she prefers, as well as how much he or she prefers it. For consumer i evaluating product j relative to product k, the preference ratio r_{ijk} is represented by d_{ik}/d_{ij}, where d_{ik} is the Euclidean distance $[(\mathbf{X}_i - \mathbf{X}_k)'(\mathbf{X}_i - \mathbf{X}_k)]^{1/2}$ between the ideal product for consumer i and the actual product k. If the ideal product is sampled independently for the two distances and the variances for any product are the same on all dimensions, the preference ratio r_{ijk} has a density function that is closely related to the density function of a doubly noncentral F distribution. If the variances for some products are not the same on all dimensions, then the density function is closely related to the density function of a ratio of quadratic forms.

Once the density functions are known, maximum likelihood methods can be used to estimate both the means and variances of the ideal and actual products. We provide a summary of the anisotropic PROSCAL model for preference ratio judgments in the subsequent section. (MacKay and Zinnes [1995] provide more detail.)

Preference ratios were chosen for the PROSCAL model because they have several desirable properties:

1. They are sensitive to differences in variance structures.
2. They are relatively easy for subjects to carry out (Birnbaum, Anderson, and Hynan 1989).
3. They are invariant and, hence, meaningful when distances are measured on a ratio scale.
4. They are unitless.

To decrease the time it takes to collect preference ratio judgments when the number of products is large, incomplete sets of judgments may be obtained from each subject. (For probabilistic models based on other types of judgments, see, for example, the articles by DeSarbo et al. 1990; De Soete, Carroll, and DeSarbo 1986; Mullen and Ennis 1991.)

Earlier versions of PROSCAL were not able to avoid the indeterminacy problem that occurs when a single ideal point is used. This was because of an independent sampling assumption that required the numerator d_{ik} and the denominator d_{ij} of the preference ratio r_{ijk} to be independent. To be independent, the subject must separately sample the values that go into the numerator and denominator of the preference ratio. In effect, this requires assuming that person i samples twice from the ideal product distribution—once to arrive at d_{ik} and once for d_{ij}.

For some products, such as the disguised products a consumer faces in a blind taste-test, the independence assumption may be appropriate. But, for undisguised stimuli, the situation that usually occurs when market structures are estimated, the independence assumption is likely to be inappropriate.[1]

MODEL DEVELOPMENT

To relax the independent sampling assumption, we begin by expressing the preference ratio r_{ijk} as a ratio of quadratic forms. Defining d_{ikn} as the difference between ideal product i and real product k on dimension n, let

$$(1) \qquad X' = (d_{ik1},\ldots,d_{ikp}, d_{ij1},\ldots,d_{ijp}),$$

which is a vector containing $2p$ elements. Making use of Equation 1, we can write d_{ik}^2/d_{ij}^2 as,

$$(2) \qquad r_{ijk}^2 = \frac{d_{ik}^2}{d_{ij}^2} = \frac{X'AX}{X'BX},$$

where

$$A = \begin{pmatrix} I_p & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$B = \begin{pmatrix} 0 & 0 \\ 0 & I_p \end{pmatrix}.$$

The variance-covariance matrix Σ_{ijk} of X depends on the sampling method used. For independent samples,

$$(3) \qquad \Sigma_{ijk} = \begin{pmatrix} \Sigma_i + \Sigma_j & 0 \\ 0 & \Sigma_i + \Sigma_k \end{pmatrix},$$

whereas for dependent samples,

$$(4) \qquad \Sigma_{ijk} = \begin{pmatrix} \Sigma_i + \Sigma_j & \Sigma_i \\ \Sigma_i & \Sigma_i + \Sigma_k \end{pmatrix}$$

Dependent sampling enables Σ_i and Σ_j to be uniquely defined. With independent sampling, this is a problem that must be solved by imposing side condition, because,

$$(5) \qquad \Sigma_i + \Sigma_j = \Sigma^*_i + \Sigma^*_j = (\Sigma_i + c) + (\Sigma_j - c).$$

Managerially, the dependent sampling model makes it possible for us to distinguish consumers' uncertainty about their ideal products from their uncertainty about the nature of the products themselves. (Dependent sampling has been proposed by Coombs, Greenberg, and Zinnes [1961] and Mullen and Ennis [1991]. However, the relation of dependent sampling to indeterminacy has, to our knowledge, never been discussed.)

To obtain maximum likelihood estimates of the location and variance parameters, it is necessary to derive the density function $f(\cdot)$ of r_{ijk}. Distribution and density functions of quadratic forms have been previously used in the multidimensional scaling (MDS) of dissimilarities (MacKay 1989). The distribution function of a ratio r^2 of quadratic forms can be expressed as the distribution of a quadratic form by making use of the property,

$$(6) \qquad F(r^2) = P\left(\frac{X'AX}{X'BX} \le r^2\right) = P(X'(A - Br^2)X \le 0) = P(X'GX \le 0),$$

where $G = A - Br^2$.

Direct calculation of the density function of r in Equation 6 is not computationally practical, but the area is seeing steady development in the mathematical literature. Difficulties center around the fact that the quadratic form in Equation 6 is noncentral (i.e., the means are non-zero), indefinite (i.e., negative eigenvalues are involved in the solution), and dependent. However, numerical procedures for calculating the distribution function of r in Equation 6 are available (for reviews, see Johnson and Kotz 1970; Mathai and Provost 1992). The density function of r^2 is numerically estimated by taking central differences of the distribution function. Differentiating to get the density function of r, we find,

$$(7) \qquad f(r) = \frac{r}{\delta}[P(X'UX \le 0) - P(X'VX \le 0)],$$

where

$U = (A - Br^2 - B\delta)$,
$V = (A - Br^2 + B\delta)$, and
2δ = the magnitude of the central difference.

The likelihood function of this unfolding model is then,

$$(8) \qquad L_u = \prod_{ijk} f(r_{ijk}).$$

Estimates of the location and variance parameters are not unique. As is true with most probabilistic geometric models, the likelihood function is invariant under a shift of the axes. It is also invariant when the mean and standard deviation estimates are multiplied by a positive constant. However, the rotational indeterminacy associated with many MDS models is avoided because of the specification of an anisotropic variance structure that fixes the orientation of the configuration.

We begin the estimation of the model by obtaining initial estimates, using the isotropic space procedure described in Zinnes and MacKay's (1987) study. Maximum likelihood estimation proceeds by reestimating (1) the variances holding the location estimates fixed and then (2) the location estimates holding the variances fixed. We repeat this two-step process until convergence occurs—usually within three iterations. The use of this alternating estimation procedure alleviates the bilateral indeterminacy of scaling (Kruskal 1978) and results in solutions that are as good as or better than those obtained in single-stage approaches. To calculate the distribution functions of Equation 7, we use Imhof's (1961) method. MacKay and Zinnes (1995) provide a detailed account of the estimation process, as well as some Monte Carlo analyses for a multiple ideal point model with independent sampling. Similar results appear to hold for the single ideal point model with dependent sampling that we propose here.

The anisotropic unfolding model, with independent and dependent sampling assumptions, has been incorporated into the PROSCAL (MacKay and Zinnes 1991) family of probabilistic scaling programs. Like the other models in this package, the user has an opportunity to specify a wide number of model variations that are distinguished by the type and number of constraints they place on the parameters

[1]The wandering ideal point model of De Soete, Carroll, and DeSarbo (1986) assumes dependent sampling, but requires zero-valued variances for all of the products. DeSarbo and colleagues (1990) do not specify product variances either. The wandering vector model of De Soete and Carroll (1983) also assumes dependent sampling.

being estimated. A discussion of many of these models can be found in MacKay's (1983) study.

MODEL EVALUATION

The proposed single ideal point model can be evaluated at many levels. The simplest, perhaps, is the evaluation suggested in the previous section. Using only preference data, can the proposed model distinguish products that are bilateral from products that are unilateral relative to the single ideal point? To answer this question, two hundred preference ratio judgments were simulated with dependent sampling for the bilateral condition {1b,I,2} shown in Figure 1. The likelihoods of the data arising under bilateral {1b,I,2} and unilateral {I,1u,2} conditions were calculated. As expected, the log-likelihood (–399) for the bilateral parameters was higher than the log-likelihood (–1211) for the unilateral parameters. This differed from the independent sampling model, in which both likelihoods were the same.

The preceding preference ratio judgments were also evaluated with a multidimensional unfolding model that assumed dependent sampling. The multidimensional unfolding program determined its own initial configuration—the configuration used at the start of the iterative process that maximizes the likelihood function—and was given no information about the bilateral nature of the parameters. The results were that the correct bilateral condition was obtained. Even when the unilateral condition parameters were used as the initial configuration, the maximum likelihood estimates that emerged were bilateral.

Sensitivity to Variances

The bilateral and unilateral conditions can be distinguished, because the density functions of the distance ratio d_{11}/d_{12} differ from one another when dependent sampling occurs. To show this more explicitly, in Figure 2, we plot the density functions for the unilateral and bilateral points. The ratio of the two density functions is given in the bottom panel of Figure 2.

Figure 2 shows that though the modes of the two distributions are nearly the same, they have different variances. The unilateral distribution has a smaller variance than the bilateral distribution. (In a somewhat different context, this difference in the variances of the two distributions was noted earlier by Coombs, Greenberg, and Zinnes [1961].) This means that high and low distance ratios are more consistent with the bilateral condition, whereas moderate values are more consistent with the unilateral condition.

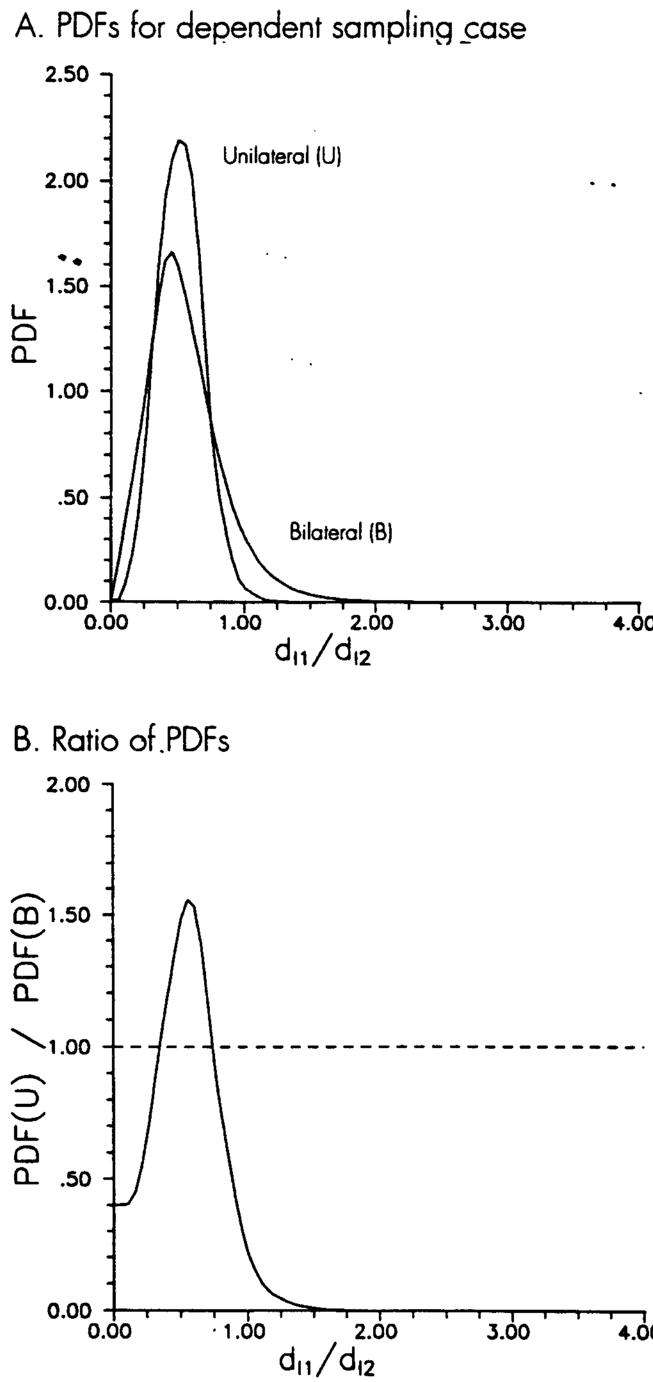
Unilateral and bilateral conditions cannot be distinguished when an independence assumption is made. When it is made, the likelihoods generated from unilateral and bilateral pairs are precisely the same.

Comparison with Nonmetric Internal Unfolding

To illustrate the relative performance of deterministic versus probabilistic models, we simulated preference ratio data. These data were based on 12 hypothetical products and one ideal product. The mean locations of the 12 products were randomly assigned within a unit circle and the ideal product mean was placed at the origin of the space. A simple isotropic Case 5 model was used and the common standard deviation of all products was set equal to .3, a value consistent with the example used in Figure 1 and the results reported in the empirical literature. Ten complete sets of distance ratios, that is, 660 judgments, were generated by calculating distance ratios from dependently sampled coordinates. Locations of the product means, labeled A through L, and the ideal product mean, labeled M, are shown in Figure 3, part A.

To provide a basis of comparison, the data were first evaluated with KYST, a popular nonmetric MDS model

Figure 2
PDFS OF THE RATIO d_{11}/d_{12}



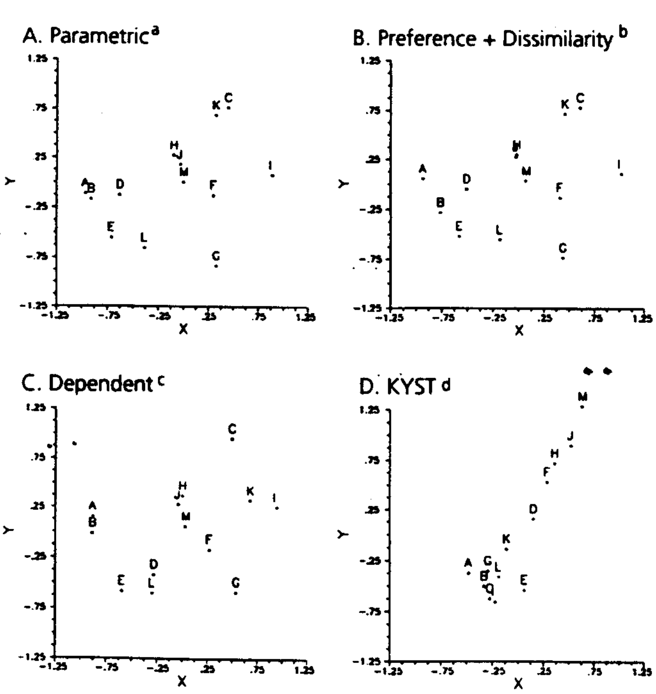
(Kruskal, Young, and Seery 1973). Being deterministic, KYST has the same problems that a probabilistic model with independent sampling has. It is unable to identify the locations of products when a single ideal point is used. In addition, it confounds variance with distance.

Because MDS estimates are sensitive to the initial configuration, ten sets of *dissimilarity* data were additionally generated among the product points of Figure 3, part A. These data were then evaluated by KYST to form the initial configuration of the twelve product points. A nonmetric, "split by rows" analysis using stress formula two was used, with very stringent convergence requirements to prevent premature termination. Despite the good start offered by the analysis of the dissimilarity data, the results of the analysis of the preference data (see Figure 3, part D) show that KYST essentially estimated a one dimensional I scale, instead of a two-dimensional J scale. All the interpoint distance information of the true configuration (Figure 3, part A) is lost. The product moment correlation of the interpoint distances among the points of Figure 3, part A with the interpoint distances among the points estimated by KYST in Figure 3, part D was –.16.

The probabilistic analysis started with a dimensionality test using both a likelihood ratio chi-square test and the minimum Akaike Information Criterion (AIC) rule (Akaike 1974). Two times the difference in the log-likelihoods is asymptotically chi-square distributed, with the degrees of

Figure 3

PARAMETRIC AND ESTIMATED CONFIGURATIONS



[a]The Parametric configuration.

[b]Joint probabilistic scaling with dependent sampling and dissimilarity data.

[c]Probabilistic scaling with dependent sampling.

[d]Deterministic nonmetric scaling with KYST.

Table 1

DIMENSIONALITY TEST STATISTICS

Dimensionality	*Number of Free Parameters*	*Log-Likelihood*	*AIC*
1	12	–1154	2332
2	25	–1120	2289
3	38	–1115	2305
4	51	–1113	2327

freedom being equal to the difference in the number of free parameters in the models being compared. When isotropic preference ratio models are estimated, the number of free parameters k is

$$k = (m + n)p + q - p - \frac{p(p-1)}{2} - 1,$$

where,

- p = the dimensionality of the space,
- m = the number of products,
- n = the number of ideal points (equal to 1 in our case), and
- q = the total number of unique variances and covariances being estimated.

The last three terms are subtracted for the centering, rotation, and scale invariance of the solution. When anisotropic models are used, the rotational invariance term should be omitted.

Table 1 shows the likelihoods, the number of free parameters, and the values of the AIC statistic for anisotropic Case 5 analyses of the simulated data in one through four dimensions. The chi-square test of the likelihoods shows that the two-dimensional model is significantly better ($p < .001$) than the unidimensional model and that the three- and four-dimensional models are *not* significantly better than the two-dimensional model ($p > .7$). However, the chi-square test must be treated cautiously, because, though a low dimensional solution is a special case of a high dimensional solution, the standard regularity condition concerning the boundary of the parameter space was not met. Specifically, setting a variance equal to zero in a low dimensionality solution violates the requirement that the true parameter vector be in the interior of the parameter space. McDonald and Xu (1992) indicate that when this regularity condition is violated, the likelihood ratio test tends to be too conservative—the results are actually more significant than they appear.

The AIC statistic is not subject to the regularity condition of the likelihood ratio test. For an unfolding model with likelihood L_u and k free parameters, AIC_k is defined by,

$$AIC_k = -2\ln L_u + 2k.$$

Using the minimum value of the AIC statistic to select the correct dimensionality gives the same result as the likelihood ratio test, namely, the two-dimensional solution.

Results from the two-dimensional probabilistic, dependent sampling model are shown in Figure 3, part C. The ideal point was correctly located in the center of the two-dimensional space. The correlation of parametric and estimated interpoint distances equals .92, which is an enormous improvement. However, some points, such as K and D, are poorly estimated.

Inclusion of Dissimilarities

The results in Figure 3, part C look good, but they can be improved. Improvement can be obtained by *jointly* estimating a product space map from both preference data and dissimilarity data. Unlike an external analysis, in which the locations of the products are fixed and the purpose of the unfolding analysis is to estimate the ideal points given the locations of the products, a joint analysis estimates the locations of the actual and ideal products *simultaneously*, as is done in an internal analysis. However, both preference and dissimilarity data are now used in the estimation process.

Joint analysis of dissimilarity and preference data is an attractive proposition, because the locations and variances of the products are common to both. Paired comparison preference judgments have a common terminus, namely, the subject. In the single ideal point model being estimated here, this means that all judgments will involve a sampling of that single ideal point. Adding dissimilarity judgments that do not involve the ideal point should lend stability to the solution and provide better parameter estimates.

The unfolding model we present allows for a dimensional dependence and a dependence between the numerator and denominator of the distance ratio. However, it assumes that the different judgments (i.e., distance ratios) are independent of one another. Comparable independence assumptions have been made earlier for a MDS model of dissimilarities (MacKay 1989; MacKay and Dröge 1990). If we also assume that the preference and dissimilarity judgments are statistically independent of each other, which is reasonable, then the joint likelihood L_j is simply,

(9) $$L_j = L_u L_d,$$

where L_u is the likelihood of the unfolding or ideal point model, defined by Equation 8, and L_d is the likelihood of the dissimilarity data, defined by,

(10) $$L_d = \prod_{jk} 2\, d_{jk} f(d_{jk}^2).$$

The function $f(d_{jk}^2)$ in Equation 10 is the density function for the squared distances between products j and k (for more detail, see MacKay 1989). The likelihood L_u is calculated over the preference data, and L_d is calculated over the dissimilarity data. (For additional comments on this approach, see Ramsay's [1980] discussion of the joint analysis of different types of judgments.)

Maximizing Equation 9 with the simulated data results in the point estimates shown in Figure 3, part B. The estimates are close to the correct values in Figure 3, part A. The correlation of the interpoint distances between the estimated and correct values is .98. Estimates of the dimensional standard deviations were .27 and .28, which is reasonably close to the correct value of .30 on each dimension.

In a real world application the results may not be so elegant: The products that a consumer judges when making dissimilarity judgments may be cognitively different from those the consumer judges when making preference judgments. Thus, consumers may base their dissimilarity judgments on a cognitive space that is unlike the one they use for preference judgments. This could happen even when the two product sets are equal. This illustrates one of the advantages of using probabilistic models. Evidence for this possibility can be statistically evaluated. If the product sets are cognitively different, the analysis can proceed with only the preference data.

APPLICATION

The market to which we apply the proposed model is the birthday gift market for young men. The gift market presents a challenging problem for any market structure model. The consumer behavior literature is rich in its description of the structural, behavioral, and motivational complexities of gift-giving (e.g., Fischer and Arnold 1990; Garner and Wagner 1991; Mick and Demoss 1990; Park 1992; Sherry 1983). Gift alternatives are often classified as examples of non-comparable products. Johnson (1984) suggests that product attributes become increasingly abstract as products become more non-comparable; and Johnson and Fornell (1987) argue that the appropriateness of continuous dimensions representing market structures increases as the degree of abstraction increases. High variances, associated with both the presence of abstract dimensions and a single ideal point model, support the use of probabilistic geometric models, such as our single ideal point model.

Data for this application came from a pilot, cross-cultural study of Japanese and American gift-giving behavior. Cross-cultural studies are attractive applications for single ideal point models, because the assumption of a common, cross-cultural, stimulus space—an assumption necessary for a multiple ideal point model—is difficult to maintain. The segment we use is of the Japanese who give birthday gifts to young men who are under 30 years of age and are their close friends (not business acquaintances). Forty-six subjects from a convenience sample in Tokyo provided dissimilarity and preference judgments for pairs of gift alternatives. The gifts, selected in consultation with several Japanese culture experts and listed in Figure 4, were chosen for their appropriateness in both cultures. Prices of the gifts were described as being within one thousand yen of one another.

At the beginning of the data collection task, a series of warm-up questions was used to get subjects familiar with making ratio judgments. Warm-up questions proceeded from relatively concrete (e.g., "how much bigger is one square than another") to relatively abstract (e.g., "how much more do you prefer painting A to painting B?") questions. Color pictures of the warm-up stimuli and gift alternatives were displayed to each subject on a personal computer. Graphic rating scales were used to record all responses. After the preference and dissimilarity judgments for the gift alternatives were completed, subjects were asked to describe how they made their judgments.

Probabilistic Joint Analysis

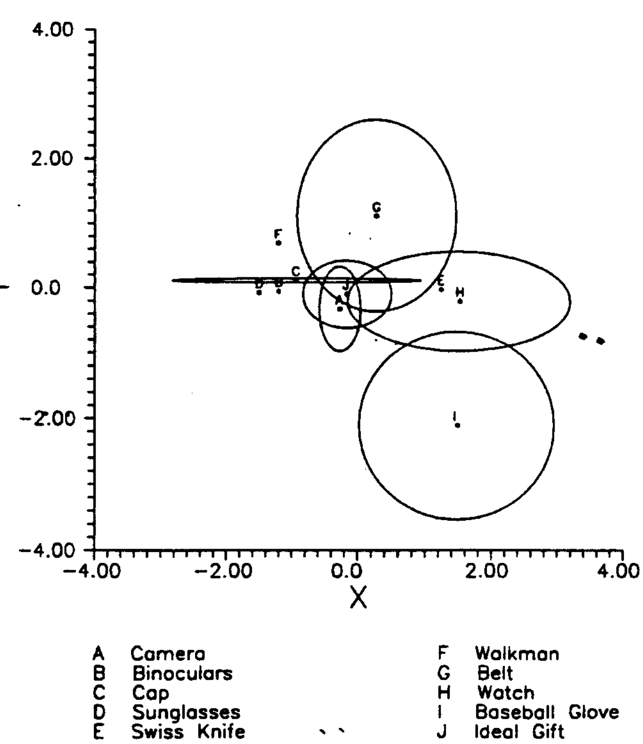
Isotropic and anisotropic, Case 3 and Case 5 models were estimated in spaces of one to three dimensions. By means of likelihood ratio tests, results using simple models were compared to those using complex models. The estimates obtained from a two-dimensional anisotropic Case 3 model were significantly ($p < .001$) better than both those obtained from a Case 3 unidimensional model and a Case 5 anisotropic two-dimensional model. The variances for five of the products—binoculars, sunglasses, Swiss army knife, Walkman, and watch—were similar. Estimates using a model that constrained the variances of these five items to be the same had a likelihood that was not significantly better when a more complex unconstrained two-dimensional model was used. The constrained two-dimensional model is portrayed in Figure 4. (Beyond two dimensions, the estimated configurations were essentially linear in their higher dimensions.)

To determine if the products subjects responded to while making dissimilarity judgments were cognitively different from those they responded to while making preference judgments, the log-likelihoods (see Table 2) were calculated. The first value, –1962, is the log-likelihood of the maximum likelihood solution for the joint analysis of the preference ratio and dissimilarity data. The second value, –1947, is the sum of the log-likelihoods of the separately scaled preference ratio and dissimilarity data. The separate scaling of the two data sets required an additional 25 free parameters to be estimated. Two times the difference in the log-likelihoods is 30, which, for 25 degrees of freedom, is not statistically significant. Thus, the separately scaled data did not fit significantly better than the jointly scaled data. This suggests that subjects perceived the products similarly in both judgment tasks. The two configurations resulting from the scaling of the preference and the combined data were extremely similar. Only one product, the Walkman, was located differently. These results are similar to those in the previous simulation.

Table 2
LOG-LIKELIHOODS FOR PREFERENCE AND DISSIMILARITY DATA

Analysis	*Log-Likelihood*
Joint Analysis	–1962
Separate Analyses	–1947

Figure 4
JOINTLY ESTIMATED CONFIGURATION AND VARIANCE STRUCTURE



Note: Standard deviational ellipses for B, D, E, and F (not shown) are identical to that for H.

With the aid of the subjects' verbal descriptions of their decision processes, the X-axis measures the strength of the gift's association with a male recipient. Gift products with high values are considered more appropriate for male recipients, whereas gift products with low scale values are considered appropriate for either male or female recipients. For example (see Figure 5, point H), the picture of the watch shown to the subjects indicated that the watch was large and had a sports style. The belt (G) of braided leather was also styled for men. The Y-axis measures the activity level involved in using the product. Products with low values, such as the baseball glove (I), are high activity products, whereas gifts with high values, such as the belt (G), are low activity products.

The ideal gift (J) for the Japanese subjects was very close to a camera (A), which is a product that involves a relatively high level of activity and is slightly more appropriate for a male recipient than for a female one. The expected distance of the baseball glove (I) from the ideal point (J) indicated that this was the least preferred gift item. The high expected distance was due to both the location and high variance of the baseball glove.

As expected, variances for the products are high. However, surprisingly, the variance for the single ideal point was relatively low, which suggests that models that ascribe all the variance to an ideal point may not be appropriate for situations such as this. The major axes of the belt (G) and camera (A) ellipses were parallel to the Y-axis. All other ellipses' major axes were parallel to the X-axis. The sum of the variances of the two-dimensions was lowest for the camera (A), the product which was closest to the single ideal. The high standard deviation on the X-axis for the cap (C) indicates that there is considerable ambiguity concerning the appropriateness of this gift for women. The high variance for the baseball glove may be because, though the Japanese love baseball, there are fewer opportunities to play baseball in Japan than in the United States.

To further evaluate the appropriateness of both the estimated variance structure and the proposed model, we estimated two additional constrained solutions. The first constrained the variances of the products to be zero—consistent with the variance structure assumption of simpler models, such as the wandering ideal point model. The second constrained the variances of the ideal point to be zero—a condition by virtue of equations 3 through 5 that is equivalent to the assumption of independent sampling. Comparisons of the two constrained model log-likelihoods to the log-likelihood of the unconstrained model showed the unconstrained model to be significantly better ($p < .001$) in both situations.

To evaluate the stability of the estimates, we split the preference ratio judgments into two equal samples. Centroids and variances were estimated for the nine real products and the single ideal product in a two-dimensional space. Forty-five interproduct expected distances and 45 interproduct distances among the centroids were calculated for each sample. (For a description of how to calculate expected distances among anisotropic distributions, see MacKay 1989.) We provide product moment correlations of these two sets of distances and present the correlation among corresponding variances and covariances in Table 3. All three correlations are high, which indicates a satisfactory degree of stability.

Deterministic External Analysis

To provide a basis for comparison, we performed an external analysis of the preference data. The dissimilarity data in this application were first evaluated nonmetrically with KYST. The resulting configuration of points for the birthday gifts was then used, in addition to the preference data, as input for PREFMAP (Carroll 1972), which is a widely used model for the external analysis of preference data (Green, Carmone, and Smith 1989).

PREFMAP involves a hierarchy of models. In the most general model, each consumer is allowed his or her own orientation and weighting of the axes. The nonmetric version of PREFMAP finds the best monotone fit of the squared distances between the actual product points and the ideal product point. The most restrictive model is a vector model. In this model, the utilities of the consumer are represented by the projections of the product points onto the consumer's vector. Analysis-of-fit statistics provided by PREFMAP show no significant difference for any pair of models, thus, indicating that the most parsimonious model, the vector model, is most appropriate. Because we are concerned with unfolding models, we present the results for both the vector model and the simple unfolding model in which the weights for all consumers are the same on all axes.

Figure 5

VECTOR AND IDEAL POINT SOLUTIONS

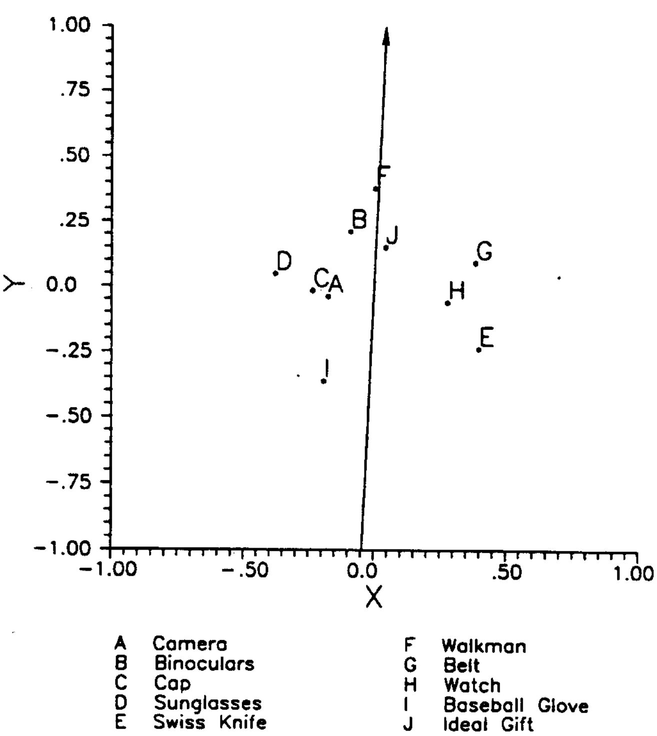


Table 3

SPLIT HALF SAMPLE CORRELATIONS

Comparison	*Correlation*
Expected Distances	.95
Centroid Distances	.97
Variances-Covariances	.96

The PREFMAP solutions are shown in Figure 5. Analysis of the similarity data by KYST prevented the collapse to a unidimensional solution that occurred when the internal unfolding model was used. The general shape of the configuration is similar to that obtained with the probabilistic model. However, the tight clustering of product centroids, {E, H} and {D, B, C}, that was observed in the probabilistic solution is less noticeable in the deterministic solution. Similar differences have been observed previously in simulation studies (Zinnes and MacKay 1983) and attributed to the confounding of distance and variation that occurs with deterministic models.

To compare the fit of the probabilistic and the deterministic analyses, a preference score S_j was defined from the preference ratio data as the mean I scale for each product. With the notation of Equation 2,

$$S_j = \frac{\sum_{i}^{m} (\prod_{k}^{n} r_{ijk})^{1/n}}{m}; \; j = 1,\ldots,n. \tag{11}$$

The correlation obtained between the preference scores and the predicted utilities from each of the three analyses is given in the first column of Table 4. Predicted utilities for the PREFMAP vector analysis were measured by the projections of the product points on the fitted vector. For the PREFMAP unfolding analysis, the predicted utilities were measured by calculating the euclidean distances between the product points and the ideal point. For the PROSCAL analysis, the predicted utilities were measured as expected distances between the product points and the ideal point. Table 4 shows that the correlation for the probabilistic analysis (.69) is larger than that of the vector model that was recommended by the PREFMAP analysis (.49). In addition, the probabilistic model correctly indicates the camera as the most preferred product, whereas the vector model estimates it as being the sixth of nine products. If, instead, we choose to use the unfolding model of PREFMAP, the fit statistics improve; the correlation rises to .51, and the rank of the camera changes to third. These comparisons indicate that the probabilistic analysis fits the preference data better than either of the PREFMAP analyses.

In the probabilistic analysis, the variances of the ideal point were relatively small. This result, though interesting and useful in some respects, presents a potential problem. If

Table 4
DESCRIPTIVE STATISTICS FOR THREE MDS MODELS

Model	*Correlation Between Mean Product Preference Scores and Estimated Utilities*	*Preference Rank Order of Most Preferred Gift*
Probabilistic Unfolding	.69	1
Deterministic Vector	.49	6
Deterministic Unfolding	.51	3

the variances of the ideal point are too small, the probabilistic model, using the assumption of dependent sampling, may fail to specify uniquely the position of the ideal point. This could happen because when the variances of the ideal point are small, there is little or no difference between the equations for the independent and dependent case. And, as we noted previously, it is not possible to solve for the ideal point when independent sampling occurs.

To explore this problem further, several simulation studies were conducted. The results, which we do not show, indicate that the variances must be extremely small—approximately one fiftieth of the values observed in this application—before the dependent sampling solution breaks down.

CONCLUSIONS

To overcome the indeterminacy associated with an internal analysis using a single ideal point, we propose using a probabilistic Thurstonian model with dependent sampling. If dissimilarity data are available, the model may be reformulated as a joint model for simultaneously scaling dissimilarity and preference judgments. Hypothesis tests can then be used to determine the appropriateness of simultaneously scaling preferences and dissimilarities.

We analyzed simulated data to determine the ability of the proposed probabilistic model to perform an internal analysis of preference data. This analysis shows that the proposed probabilistic model, in contrast to deterministic ones, can recover, with high accuracy, the parameters of the products and the ideal point.

The survey data indicate that the variances of different products are different from each other, both within and across dimensions. The existence of high variances (or levels of uncertainty) was consistent with the verbal reports of the subjects after the experiment concluded. Not only did the probabilistic model account for these different levels of uncertainty but, in doing so, it also avoided confounding distances and variances. Such confounding is typical of deterministic models. This was illustrated by analyzing the gift survey data using the deterministic model PREFMAP. The estimates provided by this deterministic model did not fit as well as the estimates from the probabilistic model.

All our empirical analyses involve products from different product categories. However, probabilistic models can also be used with brands that come from a specific product category. Previous research (MacKay and Dröge 1990) shows that there are large variances on some of the key attributes that characterize different toothpaste brands.

Changing the perceived variance of a product, be it actual or ideal, can influence market share just as strongly as changing the mean of a distribution. Both the mean and variance of a distribution are affected by marketing management decisions. In the Japanese gift market illustration, for example, the use of a gift theme when advertising large variance products, such as the belt and baseball glove, may reduce their variances, but leave the their centroids relatively unchanged. Reducing their variances will reduce their expected distance to the ideal product and their acceptability as gift items will increase. In some cases, it may be easier for a manager to increase a product's market share by taking actions that change the variance of a product's perception, than by changing its mean perceived performance on one or more attributes.

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